Maxima & Minima

ISC Class 12 Previous Years Board Questions

with Answers

2000 to Sem1-2021

1. The sum of three positive numbers is 26. The second number is 3 times as large as the first. If the sum of the squares of these numbers is least, find the numbers. [ISC 2000]

2. ABC is a right angled triangle of given area S. Find the sides of the triangle for which the area of the circumscribed circle is least. [ISC 2001]

3. An open box with a square base is to be made out of a given quantity of cardboard whose area is c^2 units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}\ units.$ [ISC 2002, 2008]

4. Three numbers are given whose sum is 180 and the ratio of the first two of them is 1 : 2. If the product of the numbers is greatest, find the numbers. [ISC 2003]

5. A closed circular cylinder has a volume of 2156 cm^3. What will be the radius of its base so that its total surface area is minimum? Find the height of the cylinder when its total surface area is minimum. [ISC 2004]

6. Prove that the area of a right angled triangle of a given hypotenuse is maximum when the triangle is isosceles. [ISC 2005]

7. Find the volume of the largest cone that can be inscribed in a sphere of radius R . [ISC 2006]

8. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$. [ISC 2007]

9. A wire of length 20 m is available to fence off a flower bed in the form of a sector of a circle. What must be the radius of the circle, if we wish to have a flower bed with the greatest possible area? [ISC 2009]

10. A closed right circular cylinder has a volume $\frac{539}{2}$ cubic units. Find the radius and the height of the cylinder so that the total surface area is minimum. [ISC 2010]

11. If the sum of the lengths of the hypotenuse and a side of a right- angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$. [ISC 2011]

12. A printed page is to have a total area of 80 sq.cm with a margin of 1 cm at the top and on each side and a margin of 1·5 cm at the bottom. What should be the dimensions of the page so that the printed area will be maximum? [ISC 2012]

13. Find the maximum volume of the cylinder which can be inscribed in a sphere of radius $3\sqrt{3}$ cm . (Leave the answer in terms of π) [ISC 2013]

14. A wire of length 50 m is cut into two pieces. One piece of the wire is bent in the shape of a square and the other in the shape of a circle. What should be the length of each piece so that the combined area of the two is minimum? [ISC 2014]

15. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius 10 cm is a square of side $10\sqrt{2}$ cm. [ISC 2015]

16. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle to get maximum area. Also, find the maximum area. [ISC 2016]

17. Show that the surface area of a closed cuboid with square base and given volume, is minimum when it is a cube. [ISC 2017]

18. A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height. [ISC 2018]

19. The volume of a closed rectangular metal box with a square base is 4096 cm^3. The cost of polishing the outer surface of the box is ₹ 4 per cm^2. Find the dimensions of the box for the minimum cost of polishing it. [ISC 2019]

20. Find the point on the straight line $2x + 3y = 6$, which is closest to the origin. [ISC 2019]

21. Show that the radius of a closed right circular cylinder of given surface area and maximum volume is equal to half of its height. [ISC 2020]

22. Prove that the area of right-angled triangle of given hypotenuse is maximum when the triangle is isosceles. [ISC 2020]

23. **Select and choose the correct answer from the following questions:** [ISC Sem 1- 2021]

A person has manufactured a water tank in the shape of a closed right circular cylinder. The volume of the cylinder is $\frac{539}{2}$ cubic units. If the height and radius of the cylinder be h and r, then:

(i) The height h in terms of radius r and the given volume will be:

(a) $h = \frac{539}{\pi r^2}$

(b) $h = \frac{539}{2\pi r^2}$

(c) $h = \frac{539}{2\pi r}$

(d) $h = \frac{539}{\pi r}$

Answer: _____

(ii) Let the total surface area of the closed cylindrical tank be S, given by

$S = \frac{539}{r} + 2\pi r^2$

If the total surface area of the tank is minimum, then the value of r will be:

(a) $r = 7\ cm$

(b) $r = 14\ cm$

(c) $r = 49\ cm$

(d) $r = \frac{7}{2}\ cm$

Answer: _____

(iii) The height of the tank h is equal to:

(a) $h = 7\ cm$

(b) $h = 14\ cm$

(c) $h = 28\ cm$

(d) $h = 2\ cm$

Answer: _____

(iv) The minimum total surface area of the tank S will be:

(a) 231 sq.cm

(b) 321 sq.cm

(c) 230 sq.cm

(d) 221 sq.cm

Answer: _____

Answer

1.	4,12,10	2.	$\sqrt{2S}, \sqrt{2S}, 2\sqrt{S}$
4.	40, 60, 80	5.	14 cm
7.	$\frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$	9.	5 cm
10.	R = 7/2 units, h = 7 units.	12.	10cm x 8 cm
13.	Maximum volume =108 π cm^3		
14.	Length for square = $\frac{200}{\pi+4}\ m$, Length for circle = $\frac{50\,\pi}{\pi+4}\ m$.		
16.	$\frac{r}{\sqrt{2}}, \sqrt{2}\,r, Area = r^2$	18.	16 cm
19.	Length = breadth = height = 16 cm	20.	$\left(\frac{12}{13}, \frac{18}{13}\right)$
23.(i)	(b) $h = \frac{539}{2\,\pi\,r^2}$	23.(ii)	(d) $r = \frac{7}{2}\ cm$
23.(iii)	(a) $h = 7\ cm$	23.(iv)	(a) 231 sq.cm

Solutions of

Maxima and Minima

ISC Class 12 Previous Years Board Questions

2000 to 2020

1. The sum of three positive numbers is 26. The second number is 3 times as large as the first. If the sum of the squares of these numbers is least, find the numbers. (ISC 2000)

Solution: Let three numbers be $x, y \; and \; z$.

$$\therefore y = 3x$$

$$and \quad x + y + z = 26$$

$$\Rightarrow \quad z = 26 - x - y$$

$$= 26 - x - 3x$$

$$= 26 - 4x$$

$$\therefore sum\ of\ their\ squares \quad S = x^2 + y^2 + z^2$$

$$= x^2 + (3x)^2 + (26 - 4x)^2$$

$$= x^2 + 9x^2 + 676 + 16x^2 - 208x$$

$$= 26x^2 - 208x + 676$$

$$Differentiating\ w.r.t.\ x, \quad \frac{dS}{dx} = 52x - 208$$

$$\Rightarrow \quad \frac{d^2S}{dx^2} = 52 > 0$$

$$\therefore sum\ of\ the\ squares\ will\ be\ least, when\ \frac{dS}{dx} = 0$$

$$\Rightarrow \quad 52x - 208 = 0$$

$$\Rightarrow \quad x = \frac{208}{52}$$

$$\Rightarrow \quad x = 4$$

$$\therefore \quad y = 3x = 12 \; and \; z = 26 - 4x = 10$$

$$Answer: \; Three\ numbers\ are\ 4, 12 \; and \; 10.$$

2. ABC is a right angled triangle of given area S. Find the sides of the triangle for which the area of the circumscribed circle is least. (ISC 2001)

Solution : Let ABC be a right angled triangle such that $\angle C = 90^0$.

The midpoint O of the hypotenuse AB will be the centre of the circumscribed circle. Let r be the radius of the circle. Then $(2r)^2 = a^2 + b^2$

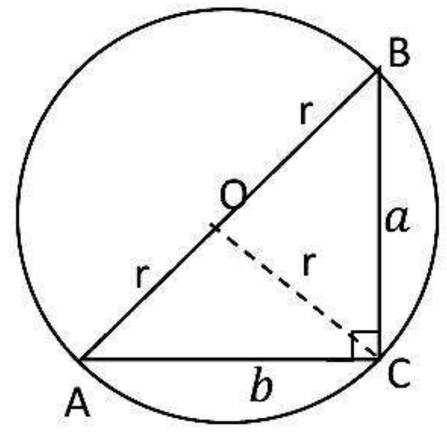
$$\Rightarrow \quad 4r^2 = a^2 + b^2$$

$$\Rightarrow \quad r = \frac{\sqrt{a^2+b^2}}{2}$$

$$Also \quad S = \frac{1}{2}ab$$

$$\Rightarrow \qquad b = \frac{2S}{a}$$

Let A denote the area of the circle.



$$A = \pi r^2$$
$$= \frac{\pi}{4}(a^2 + b^2)$$
$$= \frac{\pi}{4}\left(a^2 + \frac{4S^2}{a^2}\right)$$

Differentiating w.r.t.a, $\frac{dA}{da} = \frac{\pi}{4}\left(2a + \frac{4\,S^2(-2)}{a^3}\right)$

$$\Rightarrow \qquad \frac{dA}{da} = \frac{\pi}{4}\left(2a - \frac{8\,S^2}{a^3}\right)$$

Differentiating again $\frac{d^2A}{da^2} = \frac{\pi}{4}\left(2 + \frac{24\,S^2}{a^4}\right) > 0$

∴ For least area $\frac{dA}{da} = 0$

$$\Rightarrow \qquad \frac{\pi}{4}\left(2a - \frac{8\,S^2}{a^3}\right) = 0$$
$$\Rightarrow \qquad 2a = \frac{8S^2}{a^3}$$
$$\Rightarrow \qquad a^4 = 4S^2$$
$$\Rightarrow \qquad a^2 = 2S$$
$$\Rightarrow \qquad a = \sqrt{2S}$$
$$\therefore \qquad b = \frac{2S}{a} = \sqrt{2S}$$

$$Hypotenuse = 2r = \sqrt{a^2 + b^2}$$
$$= \sqrt{2S + 2S}$$
$$= 2\sqrt{S}$$

Answer: The lengths of the sides of the triangle are $\sqrt{2S}$, $\sqrt{2S}$ *and* $2\sqrt{S}$.

3. An open box with a square base is to be made out of a given quantity of cardboard whose area is c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units. (ISC 2008, 2002)

Solution : Let the length , breadth and height of the box be x, x and h respectively.

$$\therefore \; c^2 = x^2 + 4\,x\,h$$
$$\Rightarrow \quad h = \frac{c^2 - x^2}{4\,x}$$

Volume of the box $V = x^2\,h$

$$= x^2\left(\frac{c^2 - x^2}{4\,x}\right)$$
$$= \frac{c^2}{4}\,x - \frac{1}{4}\,x^3$$

Differentiating w.r.t. x, $\frac{dV}{dx} = \frac{c^2}{4} - \frac{3}{4}\,x^2$

Again differentiating w.r.t.x, $\frac{d^2V}{dx^2} = -\frac{6}{4}\,x < 0$

So for maximum volume, $\frac{dV}{dx} = 0$

$$\Rightarrow \quad \frac{c^2}{4} - \frac{3}{4}\,x^2 = 0$$

$$\Rightarrow \quad \frac{3}{4}\,x^2 = \frac{c^2}{4}$$

$$\Rightarrow \quad x^2 = \frac{c^2}{3}$$

$$\Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

$$\therefore \quad h = \frac{c^2 - x^2}{4\,x}$$

$$= \frac{c^2 - \frac{c^2}{3}}{4\,\frac{c}{\sqrt{3}}}$$

$$= \frac{2\,c^2}{3} \cdot \frac{\sqrt{3}}{4\,c}$$

$$= \frac{c}{2\sqrt{3}}$$

Maximum volume of the box $V = x^2\,h$

$$= \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}}$$

$$= \frac{c^3}{6\sqrt{3}}\ cubic\ units.$$ Proved.

4. Three numbers are given whose sum is 180 and the ratio of the first two of them is 1 : 2 . If the product of the numbers is greatest, find the numbers. (ISC 2003)

Solution : Given , the ratio of the first two numbers is 1 : 2.

Let the three numbers be $x, 2x\ and\ z.$

$$\therefore x + 2x + z = 180$$

$$\Rightarrow \quad z = 180 - 3x$$

Now , product of three numbers $P = x.2x.z$

$$= 2\,x^2\,z$$

$$= 2\,x^2\,(180 - 3x)$$

$$= 360\,x^2 - 6\,x^3$$

Differentiating $w.r.t.\,x,\ \frac{dP}{dx} = 720\,x - 18\,x^2$

And $\frac{d^2P}{dx^2} = 720 - 36\,x$

For maximum or minimum product, $\frac{dP}{dx} = 0$

$$\Rightarrow \quad 720\,x - 18\,x^2 = 0$$

$$\Rightarrow \quad 18\,x(40 - x) = 0$$

$$\Rightarrow \quad x = 0\ or\ x = 40$$

For $x = 0,\ \frac{d^2P}{dx^2} = 720 > 0$ $,and\ for\ x = 40,\ \frac{d^2P}{dx^2} = (720 - 36 \times 40) < 0$

$\therefore$ Product will be maximum when $x = 40$.

$\therefore$ 2nd number is 80.

And 3rd number $z = 180 - 3 \times 40 = 60.$

Answer: Three numbers are 40, 80 and 60.

5. A closed circular cylinder has a volume of 2156 cm^3. What will be radius of its base so that its total surface area is minimum? Find the height of the cylinder when its total surface area is minimum. (ISC 2004)

Solution: Let the radius and height of the cylinder be $r\ and\ h$ respectively.

$\therefore$ volume of the cylinder $V = 2156\ cm^3$

$$\Rightarrow \quad \pi r^2 h = 2156$$

$$\Rightarrow \quad h = \frac{2156}{\pi r^2}$$

Total surface area $S = 2\pi rh + 2\pi r^2$

$$= 2\pi r\left(\frac{2156}{\pi r^2}\right) + 2\pi r^2$$

$$= \frac{4312}{r} + 2\pi r^2$$

Differentiating $w.r.t.\ r,\ \frac{dS}{dr} = -\frac{4312}{r^2} + 4\pi r$

$$\therefore \quad \frac{d^2S}{dr^2} = \frac{4312 \times 2}{r^3} + 4\pi > 0$$

$\therefore$ For minimum surface area, $\frac{ds}{dr} = 0$

$$\Rightarrow \quad -\frac{4312}{r^2} + 4\pi r = 0$$

$$\Rightarrow \quad \frac{4312}{r^2} = 4\pi r$$

$$\Rightarrow \quad r^3 = \frac{4312 \times 7}{4 \times 22}$$

$$\Rightarrow \quad r = 7\ cm$$

$\therefore$ height of the cylinder $h = \frac{2156}{r^2}$

$$= \frac{2156}{49}$$

$$= 14\ cm$$

Answer: radius and Height of the cylinder are 7 cm and 14 cm respectively

when surface area is minimum.

6. Prove that the area of a right angled triangle of a given hypotenuse is maximum when the triangle is isosceles. (ISC 2005)

Solution: Let $a, b \; and \; l$ be the three sides of the right angled triangle ABC.

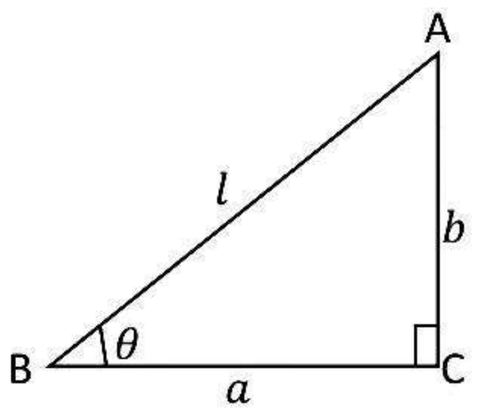
$From \; figure, \; a = l \cos\theta \quad and \quad b = l \sin\theta$

Area of the triangle $A = \frac{1}{2} a b$

$= \frac{1}{2} l \cos\theta . l \sin\theta$

$= \frac{1}{4} l^2 . 2 \cos\theta \sin\theta$

$= \frac{1}{4} l^2 \sin 2\theta$



Differentiating $w.r.t. \; \theta, \quad \frac{dA}{d\theta} = \frac{1}{2} l^2 \cos 2\theta$

And $\frac{d^2A}{d\theta^2} = -l^2 \sin 2\theta$

As θ is acute, $\sin 2\theta > 0, \quad so \quad \frac{d^2A}{d\theta^2} < 0$

$\therefore$ For maximum area of the triangle $\frac{dA}{d\theta} = 0$

$\Rightarrow \quad \frac{1}{2} l^2 \cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = 0$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\therefore a = l \cos\left(\frac{\pi}{4}\right) = \frac{l}{\sqrt{2}} \quad and \quad b = l \sin\left(\frac{\pi}{4}\right) = \frac{l}{\sqrt{2}}$

$\therefore \quad a = b = \frac{1}{\sqrt{2}}$

Therefore , the area of a right angled triangle of a given hypotenuse is maximum when the triangle is isosceles. Proved.

7. Find the volume of the largest cone that can be inscribed in a sphere of radius R. (ISC 2006)

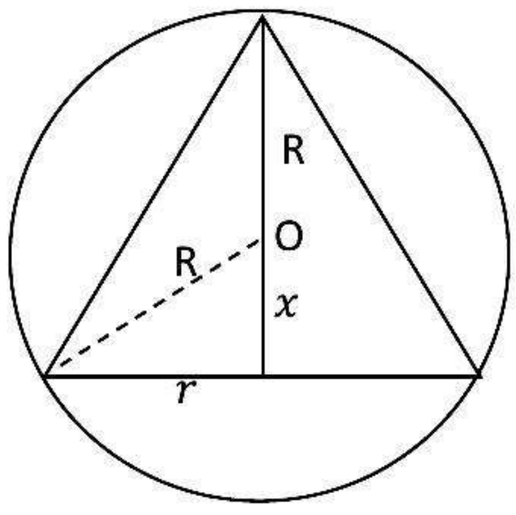
Solution : Let r be the base radius, x the distance of O, the centre of the spherefrom the base and V the volume of the cone.

Height h of the cone $= R + x$

Volume of cone $V = \frac{1}{3} \pi r^2 h$

$= \frac{1}{3} \pi (R^2 - x^2)(R + x)$

$= \frac{1}{3} \pi (R^3 + R^2 x - Rx^2 - x^3)$



Differentiating $w.r.t. \; x, \quad \frac{dV}{dx} = \frac{1}{3} \pi (R^2 - 2Rx - 3x^2)$

$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{1}{3} \pi (-2R - 6x) < 0$

$\therefore$ For maximum volume $\frac{dV}{dx} = 0$

$$\Rightarrow \quad \frac{1}{3}\pi\,(R^2 - 2Rx - 3x^2) = 0$$

$$\Rightarrow \quad R^2 - 2Rx - 3x^2 = 0$$

$$\Rightarrow \quad (R + x)(R - 3x) = 0$$

$x = -R, \frac{R}{3} \quad but \quad x \neq -R$.

V will be maximum when $x = \frac{R}{3}$

Maximum volume $V = \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right)$

$$= \frac{1}{3}\pi \cdot \frac{8R^2}{9} \cdot \frac{4R}{3}$$

$$= \frac{32}{81}\pi R^3$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$= \frac{8}{27}$ (Volume of sphere) . Answer.

8. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.
(ISC 2007)

Solution: Let l be the given slant height and θ the semi-vertical angle of the cone.

Let Radius of the base = a

and height of the cone = b

$$\therefore \quad l = \sqrt{a^2 + b^2}$$

$$\Rightarrow \quad l^2 = a^2 + b^2$$

$$\Rightarrow \quad a^2 = l^2 - b^2$$

Volume of the cone $V = \frac{1}{3}\pi\,(Radius\ of\ the\ base)^2 \times (height\ of\ the\ cone)$

$$= \frac{1}{3}\pi\, a^2\, b$$

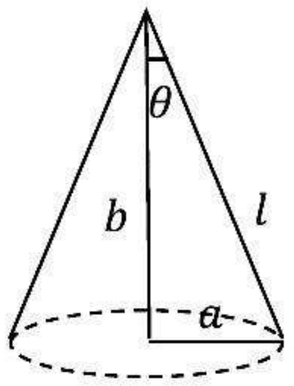
$$= \frac{1}{3}\pi\,(l^2 - b^2)b$$

$$= \frac{1}{3}\pi\,(l^2\, b - b^3)$$

$$\therefore \frac{dV}{db} = \frac{1}{3}\pi\,(l^2 - 3b^2)$$

$$\therefore \frac{d^2V}{db^2} = \frac{1}{3}\pi\,(-6b)$$

$$= -2\pi\, b \ < 0$$



For maximum volume $\frac{dV}{db} = 0$

$$\therefore \quad \frac{1}{3}\pi\,(l^2 - 3\,b^2) = 0$$

$$\Rightarrow \quad l^2 - 3\,b^2 = 0$$

$$\Rightarrow \quad 3b^2 = l^2$$

$$\Rightarrow \quad b = \frac{l}{\sqrt{3}}$$

$$\therefore \quad a^2 = l^2 - b^2$$

$$= \; l^2 - \frac{l^2}{3} = \frac{2}{3}\, l^2$$

$$\therefore \quad a = \sqrt{\frac{2}{3}}\, l$$

$$\therefore \quad \tan\theta = \frac{a}{b} = \frac{\sqrt{\frac{2}{3}}\, l}{\frac{1}{\sqrt{3}}\, l} = \sqrt{2}$$

Hence volume is maximum for $\tan\theta = \sqrt{2}$

Or $\theta = \tan^{-1}\sqrt{2}$ Proved.

9. A wire of length 20 m is available to fence of a flower bed in the form of a sector of a circle. What must be the radius of the circle, if we wish to have a flower bed with the greatest possible area? (ISC 2009)

Solution: Let O be the centre of the circle and OAB be the sector whose perimeter is 20 cm

and let $\angle$ AOB = θ

$$\therefore 2r + r\,\theta = 20$$

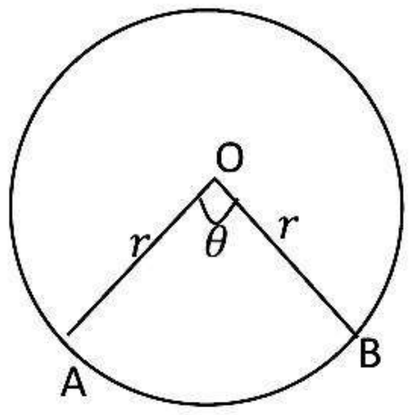
$$\Rightarrow \quad \theta = \frac{20-2r}{r}$$

Area of the sector OAB $S = \frac{\theta}{2\pi}(\pi r^2)$

$$= \frac{1}{2}\,\theta\, r^2$$

$$= \frac{1}{2}\left(\frac{20-2r}{r}\right) r^2$$

$$= 10\, r - r^2$$



$$\therefore \quad \frac{dS}{dr} = 10 - 2r$$

And $\frac{d^2S}{dr^2} = -2 < 0$

$\therefore$ For maximum area of the sector $\frac{dS}{dr} = 0$

$$\Rightarrow \quad 10 - 2r = 0$$

$$\Rightarrow \quad r = 5 \;\; cm$$ Answer

10. A closed right circular cylinder has volume $\frac{539}{2}$ cubic units. Find the radius and the height of the cylinder so that the total area is minimum. (ISC 2010)

Solution: Let r be the radius and h be the height of the cylinder.

$$\text{Volume} = \frac{539}{2} = \pi\, r^2\, h$$

$$\Rightarrow \quad h = \frac{539}{2\,\pi\, r^2}$$

Total surface area of the cylinder $S = 2\,\pi\, r\, h + 2\,\pi\, r^2$

$$= 2\pi r\left(\frac{539}{2\pi r^2}\right) + 2\pi r^2$$

$$= \frac{539}{r} + 2\pi r^2$$

Differentiating $w.r.t.r$, $\quad \frac{dS}{dr} = -\frac{539}{r^2} + 4\pi r$

Again differentiating $w.r.t.r$, $\quad \frac{d^2S}{ds^2} = \frac{539 \times 2}{r^3} + 4\pi > 0$

$\therefore$ For minimum surface area, $\quad \frac{dS}{dr} = 0$

$$\Rightarrow \quad -\frac{539}{r^2} + 4\pi r = 0$$

$$\Rightarrow \quad 4\pi r = \frac{539}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{539}{4\pi}$$

$$\Rightarrow \quad r^3 = \frac{539 \times 7}{4 \times 22}$$

$$\Rightarrow \quad r = \frac{7}{2}\ units$$ Answer.

$\therefore$ Height $\quad h = \frac{539}{2\pi r^2}$

$$= \frac{539 \times 7 \times 2 \times 2}{2 \times 22 \times 7 \times 7}$$

$$= 7\ units$$ Answer.

11. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$. (ISC 2011)

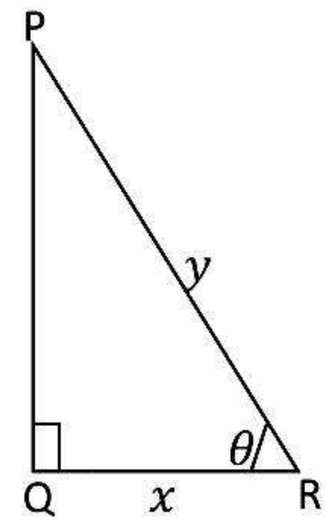
Solution : Let us consider a right-angled triangle with base x and hypotenuse y.

Let $x + y = k$ where k is a constant.

$\therefore \quad y = k - x$

Let θ be the angle between the base and hypotenuse.

Let A be the area of the triangle.



Then $\quad A = \frac{1}{2} \times QR \times PQ$

$$= \frac{1}{2}\, x\sqrt{y^2 - x^2}$$

$$= \frac{1}{2}\, x\sqrt{(k-x)^2 - x^2}$$

$$= \frac{1}{2}\, x\sqrt{k^2 + x^2 - 2kx - x^2}$$

$$= \frac{1}{2}\, x\sqrt{k^2 - 2kx}$$

$\therefore \quad A^2 = \frac{1}{4}\, x^2 (k^2 - 2kx)$

$$= \frac{k^2}{4}x^2 - \frac{k}{2}x^3$$

Differentiating $w.r.t.x$, $2A\frac{dA}{dx} = \frac{k^2}{4}.2x - \frac{k}{2}.3x^2$

$$= \frac{k}{2}x(k - 3x)$$

For maximum or minimum area, $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{k}{2}x\,(k - 3x) = 0$$

As $x \neq 0$, $x = \frac{k}{3}$

Differentiating again equation (1), $2A\frac{d^2A}{dx^2} + 2\left(\frac{dA}{dx}\right)^2 = \frac{k^2}{2} - 3kx$

Putting $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$, $2A\frac{d^2A}{dx^2} = \frac{k^2}{2} - 3k\,.\frac{k}{3}$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = -\frac{k^2}{4A} < 0$$

$\therefore A$ is maximum when $x = \frac{k}{3}$.

$\therefore y = k - \frac{k}{3} = \frac{2k}{3}$

$\therefore \cos\theta = \frac{\frac{k}{3}}{\frac{2k}{3}} = \frac{1}{2}$,

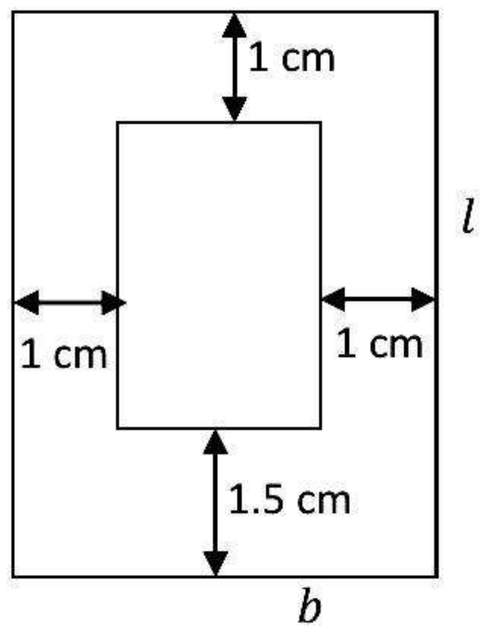
$\Rightarrow \theta = \frac{\pi}{3}$ when the area is maximum. Proved.

12. A printed page is to have a total area of 80 sq. cm. with a margin of 1 cm at the top and on each side and a margin of 1.5 cm at the bottom. What should be the dimensions of the page so that the printed area will be maximum? (ISC 2012)

Solution: Let l be the length and b be the breadth of the printed page.

$\therefore \quad l \times b = 80\ cm^2$

$\Rightarrow \quad b = \frac{80}{l}\ cm$



$\therefore$ length of the printed area $= l - 1 - 1.5 = (l - 2.5)$cm

$\therefore$ breadth of the printed area $= b - 1 - 1 = (b - 2)$ cm

$\therefore$ printed area $A = (l - 2.5)(b - 2)$

$$= lb - 2.5\,b - 2\,l + 5$$

$$= 80 - 2.5 \times \frac{80}{l} - 2\,l + 5$$

$$= (85 - 2l - \frac{200}{l})\ \text{cm}^2$$

Differentiating $w.r.t.\,l$, $\frac{dA}{dl} = -2 + \frac{200}{l^2}$

$$\Rightarrow \quad \frac{d^2A}{dl^2} = -\frac{400}{l^3} < 0$$

$\therefore$ for maximum printed area , $\frac{dA}{dl} = 0$

$$\Rightarrow \quad -2 + \frac{200}{l^2} = 0$$

$$\Rightarrow \quad 2 = \frac{200}{l^2}$$

$$\Rightarrow \quad l^2 = 100$$

$$\Rightarrow \quad l = 10\ cm$$

$$\therefore \quad b = \frac{80}{l} = \frac{80}{10} = 8\ cm$$

Hence, dimensions of printed area are 10 cm and 8 cm. Answer

13. Find the maximum volume of the cylinder which can be inscribed in a sphere of radius $3\sqrt{3}\ cm$.
(leave the answer in terms of π.) (ISC 2013)

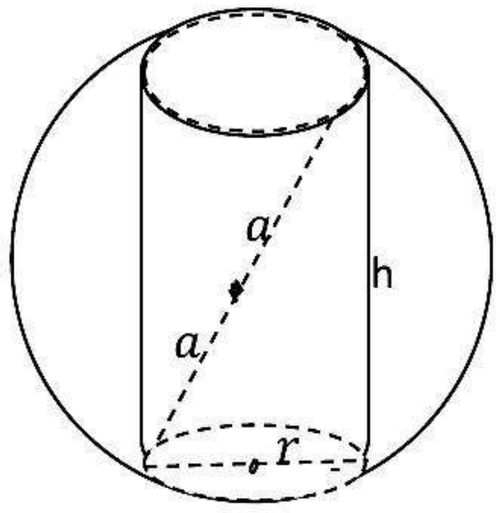
Solution: Given, radius of the sphere $= a = 3\sqrt{3}\ cm.$

Let h be the height and r be the radius of the base of the inscribed cylinder.

Then $(2r)^2 + h^2 = (2a)^2$

$$\Rightarrow \quad 4r^2 + h^2 = 4a^2$$

$$\Rightarrow \quad r^2 = \frac{4a^2 - h^2}{4}$$



$\therefore$ Volume of the cylinder $V = \pi r^2 h$

$$= \pi\left(\frac{4a^2 - h^2}{4}\right)h$$

$$= \frac{\pi}{4}(4a^2h - h^3)$$

Differentiating $w.r.t.\ h$, $\frac{dV}{dh} = \frac{\pi}{4}(4a^2 - 3h^2)$

$$\Rightarrow \quad \frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) < 0$$

$\therefore$ For maximum volume $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \frac{\pi}{4}(4a^2 - 3h^2) = 0$$

$$\Rightarrow \quad 3h^2 = 4a^2$$

$$\Rightarrow \quad h^2 = \frac{4a^2}{3}$$

$$= \frac{4 \times 27}{3}$$

$$= 36$$

$$\therefore \quad h = 6\ cm$$

$$\therefore \quad r^2 = \frac{4a^2 - h^2}{4}$$

$$= \frac{4 \times 27 - 36}{4}$$

$$= 18$$

$\therefore$ Maximum volume $V = \pi r^2 h$

$$= \pi \times 18 \times 6$$

$$= 108\pi\ cm^3$$ Answer.

14. A wire of length 50 m is cut into two pieces. One piece of the wire is bent in the shape of a square and the other in the shape of a circle. What should be the length of each piece so that the combined area of the two is minimum? (ISC 2014)

Solution: Let each side of the square be a metre and the radius of the circle be r metre.

According to the problem, $4a + 2\pi r = 50$

$$\Rightarrow \quad 2a + \pi r = 25$$

$$\Rightarrow \quad a = \frac{25 - \pi r}{2}$$

Let A be the combined area of the square and the circle.

$$A = a^2 + \pi r^2$$

$$= \left(\frac{25 - \pi r}{2}\right)^2 + \pi r^2$$

Differentiating $w.r.t.r$, $\frac{dA}{dr} = 2\left(\frac{25 - \pi r}{2}\right)\left(-\frac{\pi}{2}\right) + 2\pi r$

$$= 2\pi r - \frac{\pi}{2}(25 - \pi r)$$

Again differentiating $w.r.t.r$, $\frac{d^2A}{dr^2} = 2\pi - \frac{\pi}{2}(-\pi)$

$$= 2\pi + \frac{\pi^2}{2} > 0$$

$\therefore$ For minimum combined area, $\frac{dA}{dr} = 0$

$$\Rightarrow \quad 2\pi r - \frac{\pi}{2}(25 - \pi r) = 0$$

$$\Rightarrow \quad 2r - \frac{1}{2}(25 - \pi r) = 0$$

$$\Rightarrow \quad \frac{4r - 25 + \pi r}{2} = 0$$

$$\Rightarrow \quad 4r + \pi r = 25$$

$$\Rightarrow \quad r(4 + \pi) = 25$$

$$\Rightarrow \quad r = \frac{25}{\pi + 4}$$

$\therefore$ Length of the piece of the wire bent into the form of square $= 4a$

$$= 4\left(\frac{25 - \pi r}{2}\right)$$

$$= 50 - 2\pi\left(\frac{25}{\pi + 4}\right)$$

$$= \frac{200}{\pi + 4}\ m. \quad \text{Answer.}$$

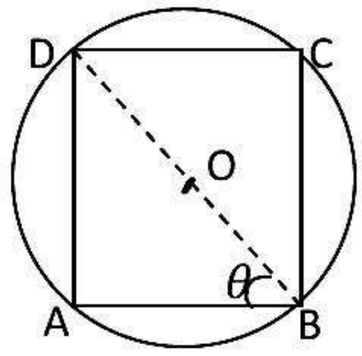
The length of the piece of wire bent in the form of circle $= 2\pi r$

$$= 2\pi\left(\frac{25}{\pi + r}\right)$$

$$= \frac{50\pi}{\pi + 4}\ m \quad \text{Answer}$$

15. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius 10 cm is a square of side $10\sqrt{2}\ cm$. (ISC 2015)

Solution : Let ABCD be a rectangle inscribed in a circle of radius 10 cm with centre at O,



then DB = 20 cm.

Let $\angle OBA = \theta$ (in radian measure)

$$0 < \theta < \frac{\pi}{2}$$

Then $AB = 20\ \cos\theta$ $\quad and \quad$ $AD = 20\ \sin\theta$

Let P be the perimeter of the rectangle ABCD.

$\therefore$ P = 2(AB + AD)

$$= 2\ (20\ \cos\theta + 20\ \sin\theta)$$

$$= 40(\cos\theta + \sin\theta)$$

Differentiating $w.r.t.\theta$, $\frac{dP}{d\theta} = 40(-\sin\theta + \cos\theta)$

$$\Rightarrow \quad \frac{d^2P}{d\theta^2} = 40\ (-\cos\theta - \sin\theta)$$

$$= -40(\cos\theta + \sin\theta) < 0$$

$\therefore$ For maximum perimeter, $\frac{dP}{d\theta} = 0$

$$\Rightarrow \quad 40(-\sin\theta + \cos\theta) = 0$$

$$\Rightarrow \quad \tan\theta = 1$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$\therefore$ P is maximum when $AB = 20\ \cos\frac{\pi}{4} = \frac{20}{\sqrt{2}} = 10\sqrt{2}\ cm$

And $AD = 20\ \sin\frac{\pi}{4} = \frac{20}{\sqrt{2}} = 10\sqrt{2}\ cm$, i.e., AB = AD

i.e., when rectangle is a square of side $10\sqrt{2}\ cm$, perimeter is maximum. Proved

16. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle to get maximum area. Also, find the maximum area. (ISC 2016)

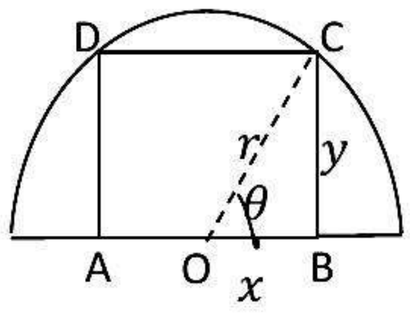
Solution : Let ABCD be a rectangle of length $2x$ and breadth y, inscribed in a semicircle of radius r and centre O. Let $\angle BOC = \theta$.

Then $\frac{y}{r} = \sin\theta$ $\quad and \quad$ $\frac{x}{r} = \cos\theta$

$\therefore$ Area of the rectangle $A = 2xy$

$$= 2\ r\cos\theta\ .r\sin\theta$$

$$= r^2\ \sin 2\theta$$



Differentiating $w.r.t.\theta$, $\frac{dA}{d\theta} = 2\ r^2\ \cos 2\theta$

Again differentiating, $\frac{d^2A}{d\theta^2} = -4\ r^2\ \sin 2\theta$

For maximum or minimum area, $\frac{dA}{d\theta} = 0$

$$\Rightarrow \quad 2r^2 \cos 2\theta = 0$$

$$\Rightarrow \quad \cos 2\theta = 0 \qquad as\ r \neq 0$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

For $\theta = \frac{\pi}{4}, \frac{d^2A}{d\theta^2} = -4r^2 \sin\frac{\pi}{2} < 0$

$\therefore$ Area will be maximum when length $2x = 2r \cos\frac{\pi}{4} = \frac{2r}{\sqrt{2}} = r\sqrt{2}\ units$. Answer.

and breadth $y = r \sin\frac{\pi}{4} = \frac{r}{\sqrt{2}} = \frac{r\sqrt{2}}{2}\ units.$ Answer

$\therefore$ Maximum area of the rectangle $= 2xy = r\sqrt{2}.\frac{r\sqrt{2}}{2} = r^2\ sq.units.$ Answer.

17. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube. (ISC 2017)

Solution : Let V be the fixed volume of a closed cuboid with length a, breadth a and height h.

Let S be its surface area.

$\therefore \quad V = a^2 h$

$\Rightarrow \quad h = \frac{V}{a^2}$

$\therefore \quad S = 2(a^2 + ah + ah)$

$= 2a^2 + 4ah$

$= 2a^2 + \frac{4V}{a}$

Differentiating $w.r.t.\ a$, $\frac{dS}{da} = 4a - \frac{4V}{a^2}$

Differentiating again $\frac{d^2S}{da^2} = 4 + \frac{8V}{a^3} > 0$

$\therefore$ For minimum surface area, $\frac{dS}{da} = 0$

$\Rightarrow \quad 4a - \frac{4V}{a^2} = 0$

$\Rightarrow \quad a^3 = V$

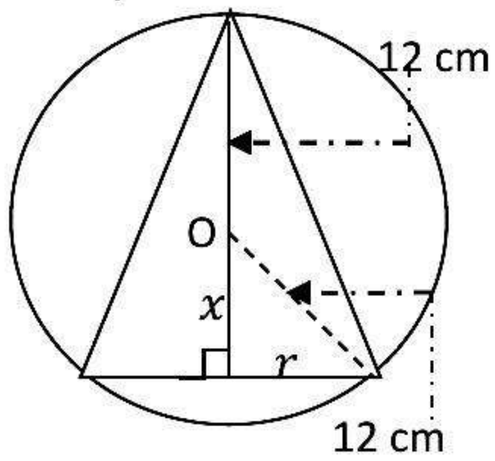
$\Rightarrow \quad a^3 = a^2 h$

$\Rightarrow \quad a = h$

So, S is minimum when length $= a$, breadth $= a$ and height $= a$, i.e., when it is a cube. Proved

18. A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height. (ISC 2018)

Solution : Let r be the base radius, x the distance of O , the centre of the sphere from the base and V the volume of the cone.



$\therefore$ Height h of the cone $= 12 + x$

Volume of the cone $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi (12^2 - x^2)(12 + x)$

$= \frac{1}{3}\pi (12^3 + 12^2 x - 12 x^2 - x^3)$

Differentiating $w.r.t.x$, $\frac{dV}{dx} = \frac{1}{3}\pi (144 - 24 x - 3 x^2)$

Differentiating again, $\frac{d^2V}{dx^2} = \frac{1}{3}\pi (-24 - 6 x) < 0$

$\therefore$ For maximum volume of the cone $\frac{dV}{dx} = 0$

$\Rightarrow \quad \frac{1}{3}\pi (144 - 24x - 3 x^2) = 0$

$\Rightarrow \quad 144 - 24 x - 3 x^2 = 0$

$\Rightarrow \quad 48 - 8 x - x^2 = 0$

$\Rightarrow \quad 48 - 12 x + 4x - x^2 = 0$

$\Rightarrow \quad 12(4 - x) + x(4 - x) = 0$

$\Rightarrow \quad (4 - x)(12 + x) = 0$

$\Rightarrow \quad x = 4, -12 \quad but \quad x \neq -12$

$\therefore$ Height of the cone when volume is maximum $h = 12 + x$

$= 12 + 4$

$= 16\ cm$ Answer.

19. The volume of a closed rectangular metal box with a square base is 4096 cm^3. The cost of polishing the outer surface of the box is ₹ 4 per cm^2. Find the dimensions of the box for the minimum cost of polishing it. (ISC 2019)

Solution : Let a be the length of each side of the square base and h be the height of the metal box.

Volume of rectangular metal box $V = 4096\ cm^3$

$\therefore \quad a^2 h = 4096$

$\Rightarrow \quad h = \frac{4096}{a^2}$

Let S be the area of the metal sheet required to construct it.

$\therefore \quad S = 2(a^2 + ah + ah)$

$= 2 a^2 + 4 ah$

$= 2 a^2 + 4a . \frac{4096}{a^2}$

$= 2 a^2 + \frac{16384}{a}$

Let C be the total cost of polishing its outer surface in rupees.

$\therefore \quad C = 4\left(2 a^2 + \frac{16384}{a}\right)$

Differentiating $w.r.t.a$, $\frac{dC}{da} = 4\left(4a - \frac{16384}{a^2}\right)$

Differentiating again, $\frac{d^2C}{da^2} = 4\left(4 + \frac{16384 \times 2}{a^3}\right) > 0$

$\therefore$ For minimum cost, $\frac{dC}{da} = 0$

$$\Rightarrow \quad 4\left(4a - \frac{16384}{a^2}\right) = 0$$

$$\Rightarrow \quad a^3 = \frac{16384}{4}$$

$$\Rightarrow \quad a^3 = 4096$$

$$\Rightarrow \quad a^3 = 2^{12}$$

$$\Rightarrow \quad a = 16\ cm$$

$$\therefore \quad h = \frac{4096}{a^2}$$

$$= \frac{4096}{16 \times 16}$$

$$= 16\ cm.$$

$\therefore$ Dimensions of the box is $16\ cm \times 16\ cm \times 16\ cm$. Answer.

20. Find the point on the straight line $2x + 3y = 6$, which is closest to the origin. (ISC 2019)

Solution : Let $P(x, y)$ be any point on the straight line $2x + 3y = 6$.

Let O (0,0) be the origin.

Now $2x + 3y = 6$

$$\therefore \quad y = \frac{6-2x}{3}$$

$$= 2 - \frac{2x}{3}$$

Then distance d = OP $= \sqrt{x^2 + y^2}$

$$= \sqrt{x^2 + \left(2 - \frac{2x}{3}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{4x^2}{9} - \frac{8x}{3}}$$

$$= \sqrt{\frac{13x^2}{9} - \frac{8x}{3} + 4}$$

Let $z = OP^2 = \frac{13x^2}{9} - \frac{8x}{3} + 4$

Then d is least when z is least.

Differentiating $w.r.t.x$, $\frac{dz}{dx} = \frac{26x}{9} - \frac{8}{3}$

Again differentiating, $\frac{d^2z}{dx^2} = \frac{26}{9} > 0$

$\therefore$ Distance will be minimum when $\frac{dz}{dx} = 0$

$$\Rightarrow \quad \frac{26x}{9} - \frac{8}{3} = 0$$

$$\Rightarrow \quad x = \frac{8 \times 9}{3 \times 26}$$

$$= \frac{12}{13}$$

$$\therefore \quad y = 2 - \frac{2}{3}x$$

$$= 2 - \frac{2}{3} \times \frac{12}{13}$$

$$= \frac{18}{13}$$

Hence the required point on the straight line $2x + 3y = 6$ which is nearest to the origin is $\left(\frac{12}{13}, \frac{18}{13}\right)$. Answer

21. Show that the radius of a closed right circular cylinder of given surface area and maximum volume is equal to half of its height. (ISC 2020)

Solution : Let r be the radius of the base and h be the height of the closed circular cylinder.

Let S be the surface area and V be the volume of the cylinder.

$\therefore \quad S = 2\pi r^2 + 2\pi r h$

$\Rightarrow \quad 2\pi r h = S - 2\pi r^2$

$\Rightarrow \quad h = \frac{S}{2\pi r} - r$

and $V = \pi r^2 h$

$\Rightarrow \quad V = \pi r^2 \left(\frac{S}{2\pi r} - r\right)$

$\Rightarrow \quad V = \frac{S}{2} r - \pi r^3$

Differentiating $w.r.t.\ r$,

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$

Again differentiating $w.r.t.\ r$,

$\frac{d^2V}{dr^2} = 0 - 6\pi r = -6\pi r < 0$

$\therefore$ For maximum volume

$\frac{dV}{dr} = 0$

$\Rightarrow \quad \frac{S}{2} - 3\pi r^2 = 0$

$\Rightarrow \quad \frac{S}{2} = 3\pi r^2$

$\Rightarrow \quad S = 6\pi r^2$

$\Rightarrow \quad 2\pi r^2 + 2\pi r h = 6\pi r^2$

$\Rightarrow \quad 2\pi r h = 4\pi r^2$

$\Rightarrow \quad h = 2r$

$\Rightarrow \quad r = \frac{h}{2}$

$\Rightarrow \quad$ Radius = $\frac{1}{2}$ × height Proved

22. Prove that the area of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles. (ISC 2005, 2020)

Solution : Let $a, b\ and\ l$ be the three sides of the right angled triangle ABC.

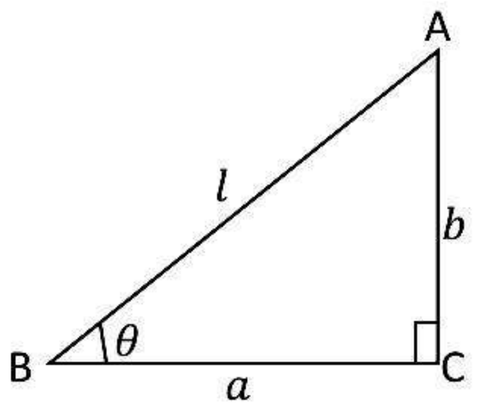
$From\ figure,\ a = l\ \cos\theta \quad and \quad b = l\ \sin\theta$

Area of the triangle $A = \frac{1}{2}\ a\ b$

$= \frac{1}{2}\ l\ \cos\theta\ .l\ \sin\theta$

$= \frac{1}{4}\ l^2.2\ \cos\theta\ \sin\theta$

$= \frac{1}{4}\ l^2\ \sin 2\theta$



Differentiating $w.r.t.\ \theta,\quad \frac{dA}{d\theta} = \frac{1}{2}\ l^2\ \cos 2\theta$

And $\frac{d^2A}{d\theta^2} = -\ l^2\ \sin 2\theta$

As θ is acute, $\sin 2\theta > 0,\quad so \quad \frac{d^2A}{d\theta^2} < 0$

$\therefore$ For maximum area of the triangle $\frac{dA}{d\theta} = 0$

$\Rightarrow \quad \frac{1}{2} l^2 \cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = 0$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\therefore a = l\ \cos\left(\frac{\pi}{4}\right) = \frac{l}{\sqrt{2}} \quad and \quad b = l\ \sin\left(\frac{\pi}{4}\right) = \frac{l}{\sqrt{2}}$

$\therefore \quad a = b = \frac{1}{\sqrt{2}}$

Therefore , the area of a right angled triangle of a given hypotenuse is maximum when the triangle is isosceles. Proved.